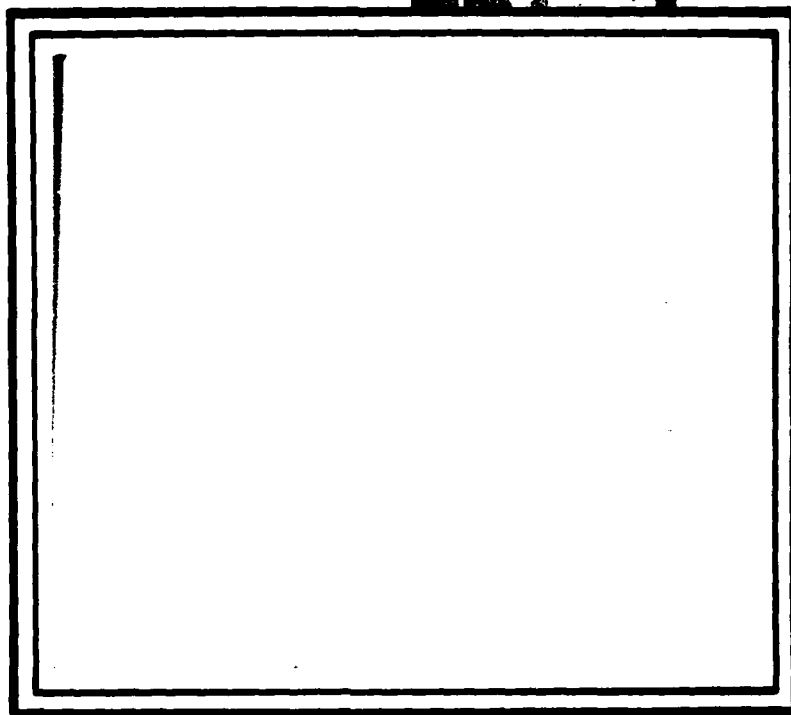END
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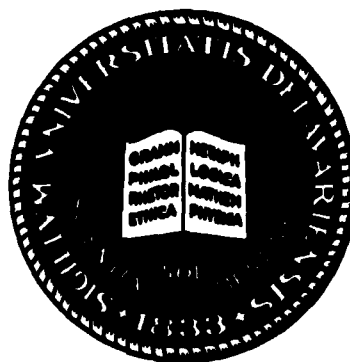
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Remarks on the Inverse Scattering
Problem for Low Frequency
Acoustic Waves*

by

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# I. Introduction

In a recent series of papers ([2], [3], [4]) the author has investigated the low frequency inverse scattering problem for "hard" and "soft" infinite cylinders and "soft" obstacles in space. It is the purpose of this paper to complete the story and consider the inverse scattering problem in space where the obstacle is "hard", i.e. the boundary condition is of Neumann type. A basic result of the analysis to date is that the low frequency inverse scattering problem can be stabilized. In the case of domains in the plane this is accomplished by transmitting two plane waves from different directions, where the second plane wave is used to determine the transfinite diameter of the obstacle ([2], [3]). Due to the lack of the availability of conformal mapping techniques, this approach does not extend to the three dimensional inverse scattering problem and alternate methods must be developed. In the case of a "soft" obstacle (i.e. Dirichlet boundary data) stability was obtained by assuming an "a priori" knowledge of the radius of a ball containing the unknown scattering obstacle in its interior and using the fact that the boundary of the scattering obstacle is a level surface of the limiting static potential problem ([4]). Since the static solution is simply the conductor potential of the obstacle irregardless of which direction the incoming wave impinges upon the obstacle, no new information is gathered by sending in plane waves from different directions (however see Section IV of this paper). In the present paper we shall be considering the inverse scattering

problem for a "hard" obstacle in space and as will be seen the situation is considerably altered. In particular, in order to achieve stability it is necessary to not only assume an "a priori" knowledge of the radius of a ball containing the scattering obstacle in its interior, but also on "a priori" bound on the gradient of the velocity potential of the total field for small values of the wave number. Further complications arise from the facts that the boundary of the obstacle is no longer a level surface of an appropirate potential problem and the solution of the limiting static potential problem is simply $u_o(\underset{\sim}{x}) \equiv 1$ which gives no information on the shape of the obstacle. These problems will be circumvented by identifying the second term in the low frequency expansion of the total field as the velocity potential of an incompressible, irrotational fluid flow past the scattering obstacle and using this function to determine the streamlines of the flow outside the given ball containing the obstacle in its interior. In this case sending in plane waves from different directions is equivalent to fluid flowing past the obstacle in different directions, and knowledge of the resulting streamlines then gives more information on the shape of the obstacle. In order to estimate the actual shape of the obstacle an important role is played by the fact that from the far field data it is possible to deduce the volume of the obstacle. Hence from a knowledge of the streamlines and the volume of the obstacle, accurate estimates can be made on its shape.

## II. Reduction of the Inverse Scattering Problem to a Nonlinear Moment Problem

The aim of this section is to reduce the inverse scattering problem for a "hard" obstacle to a nonlinear moment problem involving the velocity potential of an incompressible, irrotational fluid flow past the unknown scattering obstacle. We shall assume that the scattering obstacle D is a bounded simply connected domain containing the origin with smooth boundary $\partial D$. In order to achieve our aim it is necessary to first consider the direct scattering problem, i.e. to find a solution $u \in C^2 (R^3 \backslash \overline{D}) \cap C^1 (R^3 \backslash D)$ such that

$$u(\underset{\sim}{x}) = u^i(\underset{\sim}{x}) + u^s(\underset{\sim}{x}) \qquad \text{in } R^3 \backslash D \qquad (2.1a)$$

$$\Delta_3 u + k^2 u = 0 \qquad \text{in } R^3 \backslash \overline{D} \qquad (2.1b)$$

$$\frac{\partial u}{\partial \nu} (\underset{\sim}{x}) = 0 \qquad \text{for } \underset{\sim}{x} \in \partial D \qquad (2.1c)$$

$$\lim_{r \to \infty} r \left( \frac{\partial u^s}{\partial r} - iku^s \right) = 0 \qquad (2.1d)$$

where the "incoming wave" $u^i$ is a solution of (2.1b) in all of $R^3$, $\nu$ is the unit outward normal to $\partial D$, the wave number k is positive, and the radiation condition (2.1d) for the "scattered wave" $u^s$ is assumed to hold uniformly in all direction as $r = |\underset{\sim}{x}|$ tends to infinity. The existence of a solution to (2.1a) – (2.1d)

is well known (c.f. [9]) and our first aim is to find a low frequency approximation to u evaluated on $\partial D$. To this end we note that from [1] and [7] we can reformulate (2.1a) - (2.1d) as the integral equation.

$$u^i(\underset{\sim}{x}) + \frac{1}{4\pi} \int_{\partial D} [u(\underset{\sim}{y}) - u(\underset{\sim}{x})] \frac{\partial}{\partial \nu} \frac{1}{|\underset{\sim}{x}-\underset{\sim}{y}|} ds(y)$$

$$+ \frac{1}{4\pi} \int_{\partial D} u(\underset{\sim}{y}) \frac{\partial}{\partial \nu} \left[\frac{\exp [ik |\underset{\sim}{x}-\underset{\sim}{y}|] -1}{|\underset{\sim}{x} - \underset{\sim}{y}|}\right] ds(y) = u(\underset{\sim}{x}) \qquad (2.2)$$

for $\underset{\sim}{x} \in R^3 \backslash D$, or in obvious operator notation

$$u^i + \underset{\sim}{L}_k [u] = u. \qquad (2.3)$$

The important point here is that the spectral radius of $\underset{\sim}{L}_o$ is less than one and that

$$|| \underset{\sim}{L}_k - \underset{\sim}{L}_o || = 0(k^2) \qquad (2.4)$$

where $|| \cdot ||$ denotes the maximum operator norm ([7]). Hence for k sufficiently small, (2.3) can be solved by successive approximations and $|| (\underset{\sim}{I} - \underset{\sim}{L}_k)^{-1} - (\underset{\sim}{I} - \underset{\sim}{L}_o)^{-1} || = 0(k^2)$ ([8], p. 164). In particular for $u^i(\underset{\sim}{x}) = e^{ik x} 1, \underset{\sim}{x} = (x_1, x_2, x_3)$, and k sufficiently small, we have for $\underset{\sim}{x} \varepsilon \partial D$ that

$$u(x) = \sum_{n=0}^{\infty} \underset{\sim}{L_k^n} [1 + i\, kx_1] + 0(k^2)$$

$$= \sum_{n=0}^{\infty} \underset{\sim}{L_o^n} [1 + i\, kx_1] + 0(k^2) \qquad (2.5)$$

$$= 1 + ik\, u_o(x) + 0(k^2)$$

where $u_o$ can be identified as the unique harmonic function defined in the exterior of D such that

$$u_o(x) = x_1 + u_o^s(x) \qquad \text{in } R^3 \backslash D \qquad (2.6a)$$

$$\Delta_3\, u_o = 0 \qquad \text{in } R^3 \backslash \overline{D} \qquad (2.6b)$$

$$\frac{\partial u}{\partial \nu}(x) = 0 \qquad \text{for } x \varepsilon \partial D \qquad (2.6c)$$

$$\lim_{r \to \infty} u_o^s(x) = 0, \qquad (2.6d)$$

i.e. $u_o$ is the velocity potential for an incompressible, irrotational fluid flow past D having constant velocity in the $x_1$ direction at infinity.

We now proceed to the reduction of the inverse scattering problem to a nonlinear moment problem. From (2.2) we have that ([5], p. 316)

$$u^s(x) = \frac{1}{4\pi} \int_{\partial D} u(y)\, \frac{\partial}{\partial \nu} \left[ \frac{\exp[ik|x-y|]}{|x-y|} \right] ds(y)$$

$$= \frac{e^{ikr}}{r} \, F(\underset{\sim}{\eta};k) + 0 \, (\frac{1}{r^2}); \; \underset{\sim}{\eta} = \frac{\underset{\sim}{x}}{|\underset{\sim}{x}|} \qquad (2.7)$$

where

$$F(\underset{\sim}{\eta};k) = \frac{1}{4\pi} \int_{\partial D} u(\underset{\sim}{y}) \, \frac{\partial}{\partial \nu} \, \exp \, [-ik\underset{\sim}{\eta} \cdot \underset{\sim}{y}] \, ds(y) \qquad (2.8)$$

The inverse scattering problem we are considering is to determine D from a knowledge of the far field pattern F for all directions $\underset{\sim}{\eta}$ and low values of the wave number k. To this end we write $\underset{\sim}{\eta}$ and $\underset{\sim}{y}$ in spherical coordinates as

$$\underset{\sim}{\eta} = (\sin \theta \cos \phi, \, \sin \theta \sin \phi, \, \cos \theta) \qquad (2.9)$$

$$\underset{\sim}{y} = \rho(\sin \theta' \cos \phi', \, \sin \theta' \sin \phi', \, \cos \theta')$$

and expand F in a series of spherical harmonics

$$F(\underset{\sim}{\eta};k) = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} a_{mn}(k) \, P_n^m \, (\cos \theta) \, e^{im\phi}. \qquad (2.10)$$

Then we can easily deduce from (2.8), (2.10) and the relation (c.f. [6])

$$e^{-ik\underset{\sim}{\eta}\cdot\underset{\sim}{y}} = \sqrt{\frac{\pi}{2k\rho}} \sum_{n=0}^{\infty} (-i)^n (2n+1) J_{n+\frac{1}{2}}(k\rho) P_n(\cos\gamma)$$

(2.11)

$$\underset{\sim}{\eta}\cdot\underset{\sim}{y} = \rho \cos\gamma$$

that

$$a_{nm}(k) = \sqrt{\frac{\pi}{2}}\, i^{-n} \int_{\partial D} u(\underset{\sim}{y}) \frac{\partial}{\partial\nu} \left[ \frac{1}{\sqrt{k\rho}}\, J_{n+\frac{1}{2}}(k\rho) \cdot \right.$$

$$\left. P_n^m(\cos\theta')e^{-im\phi'} \right] ds(y).$$

(2.12)

where we have used the standard notation for associated Legendre polynomials, Legendre polynomials, and Bessel functions. Since

$$\frac{1}{\sqrt{k\rho}}\, J_{n+\frac{1}{2}}(k\rho) = \frac{(k\rho)^n}{2^{n+\frac{1}{2}}\, \Gamma(n+\frac{3}{2})} + 0\,(k\rho)^{n+2}$$

(2.13)

we have from (2.12) and (2.13) that for $n \geq 1$

$$a_{nm}(k) = \frac{i^{-n}\sqrt{\pi}\, k^n}{2^{n+1}\, \Gamma(n+\frac{3}{2})} \int_{\partial D} u(\underset{\sim}{y}) \frac{\partial}{\partial\nu} \left[ \rho^n\, P_n^m(\cos\theta')e^{-im\phi'} \right] ds(y)$$

(2.14)

$$+ 0\,(k^{n+2})$$

and since the quantity in brackets is harmonic, (2.5) and (2.14) imply that

$$a_{nm}(k) = \frac{i^{-n+1}\sqrt{\pi}\,k^{n+1}}{2^{n+1}\,\Gamma(n+\frac{3}{2})} \cdot \int_{\partial D} u_0(\underset{\sim}{y})\,\frac{\partial}{\partial\nu}\left[\rho^n\,P_n^m(\cos\theta')\,e^{-im\phi'}\right]ds(y)$$

$$+ 0(k^{n+2}) \tag{2.15}$$

for $n\geq1$. In the case $n=0$ a similiar analysis shows that

$$a_{oo}(k) = -\frac{\sqrt{\pi}\,k^2}{8\Gamma(\frac{5}{2})}\int_{\partial D}\frac{\partial}{\partial\nu}\,\rho^2\,ds + 0(k^3). \tag{2.16}$$

If we now define

$$\mu_{nm} = \lim_{k\to o}\left[\frac{2^{n+1}\Gamma(n+\frac{3}{2})}{i^{-n+1}\sqrt{\pi}k^{n+1}}\,a_{nm}(k)\right]\ ;\ n\geq1$$

$$\mu_{oo} = -\lim_{k\to o}\left[\frac{4\Gamma(\frac{5}{2})}{3k^2}\,a_{oo}(k)\right] \tag{2.17}$$

then we can rewrite (2.15) in the form

$$\mu_{nm} = \int_{\partial D}u_0(\underset{\sim}{y})\,\frac{\partial}{\partial\nu}\left[\rho^n\,P_n^m(\cos\theta')e^{-im\phi'}\right]ds(y)\ ;\ n\geq1 \tag{2.18}$$

and note that (2.16) yields

$$\mu_{oo} = \frac{1}{6}\int_{\partial D}\frac{\partial}{\partial\nu}\,\rho^2\,ds$$

$$= \frac{1}{6}\iint_D\Delta_3\rho^2\,dv \tag{2.19}$$

$$= \iint_D dv$$

$$= \text{volume of } D.$$

Hence we have reduced our inverse scattering problem to that of determining D from the nonlinear moment problem (2.18) along with the fact that we now know the volume of D from the far field data $\mu_{oo}$. Note that although the $\mu_{nm}$ are explicitly computable from the far field pattern F, small errors in measuring the coefficients $a_{nm}(k)$ will result in large errors in the numbers $\mu_{mn}$ if n is large. Hence in practice we must assume that only a finite number of the $\mu_{nm}$ are known and for the sake of simplicity we shall assume that these are known exactly. Observe that the "moment" problem (2.18) is nonlinear since both $u_o$ and the region of integration depend on D.

## III. Analysis of the Nonlinear Moment Problem

We shall now show how information on the shape of D can be extracted from the nonlinear moment problem (2.18). First suppose that the moments $\mu_{nm}$ are known for n = 1, 2...N, $-n \le m \le n$, and that constants a and M are known such that

(1) $D \subset B = \{\underset{\sim}{x} : |\underset{\sim}{x}| < a\}$

(2) $\max_{R^3 \setminus B} |\operatorname{grad} u| \le kM$ for $0 \le k \le k_o$

where $k_o$ is a fixed constant and M is independent of k. Note that from (2.5) we have from the second condition that

$$\max_{R^3 \setminus B} |\operatorname{grad} u_o| \le M. \tag{3.1}$$

For $\underline{x} \in \partial B$ we have that $u_0$ has an expansion of the form

$$u_o(\underline{x}) = x_1 + \sum_{n=0}^{\infty} \sum_{m=-n}^{n} b_{mn}\rho^{-n-1}P_n^m(\cos\theta)e^{im\phi}. \qquad (3.2)$$

Furthermore from Green's formula, (2.6c) and (2.18) we have that for $n \geq 1$

$$\mu_{nm} = \int_{\partial B} \{u_o \frac{\partial}{\partial\rho} [\rho^n P_n^m(\cos\theta')e^{-im\phi'}]$$
$$- \frac{\partial u_o}{\partial\rho}[\rho^n P_n^m(\cos\theta')e^{-im\phi'}]\}ds \qquad (3.3)$$

and from (3.2), (3.3) and the orthogonality of the associated Legendre polynomials we can conclude that

$$\mu_{nm} = 4\pi \, b_{nm} \frac{(n+m)!}{(n-m)!} \; ; \quad n \geq 1 \, . \qquad (3.4)$$

For $n=0$ we have from Green's formula that

$$0 = -\int_{\partial B} \frac{\partial u_o}{\partial\rho} \, ds$$
$$= b_{oo} \int_0^{2\pi}\int_0^{\pi} \sin\theta d\theta d\phi \qquad (3.5)$$
$$= 4\pi \, b_{oo},$$

i.e., $b_{oo} = 0$. Hence since the $\mu_{nm}$ are assumed known for $n=1,2,\ldots,N,-n \leq m \leq n$, we know

$$u_o^N(\underline{x}) = x_1 + \sum_{n=0}^{N} \sum_{m=-n}^{n} b_{nm}\rho^{-n-1}P_n^m(\cos\theta)e^{im\phi} \qquad (3.6)$$

and from assumptions 1) and 2) above we can easily obtain explicit $L_2$ error estimates for grad $(u_0 - u_0^N)$ in terms of the constants a, M and N. In particular if $x = (x_1, x_2, x_3)$ then $\frac{\partial}{\partial x_i}(u_0 - u_0^N)$, i=1,2,3, is harmonic and has an expansion of the form

$$\frac{\partial}{\partial x_i}(u_0 - u_0^N) = \sum_{n=N+1}^{\infty} \sum_{m=-n}^{n} c_{nm}^{(i)} \rho^{-n-1} P_n^m(\cos\theta) e^{im\phi}. \quad (3.7)$$

Hence we can conclude by a short calcuation (c.f. [4]) that

$$\frac{1}{4\pi\rho^2} \int_{|x|=\rho} |\,\mathrm{grad}(u_0 - u_0^N)\,|^2 ds \leq 6\pi M \sum_{n=N+1}^{\infty} (2n+1)\left(\frac{a}{\rho}\right)^{2n+2} \quad (3.8)$$

and thus we can determine how many Fourier coefficients of the far field pattern are needed in order to approximate grad $u_0$ for $|x| \geq a_0 > a$ to obtain a given degree of accuracy in the $L_2$ sense.

Identifying $u_0$ as the velocity potential of an incompressible, irrotational fluid flow past D, we now note that the unknown obstacle D is a surface containing streamlines of the fluid flow past D. Hence if we can determine streamlines lying in the exterior of D this will give us information on the shape of D. To this end let $x_0$ be a point in $\mathbb{R}^3 \backslash B$. Then the streamline passing through $x_0$ can be determined by solving the system of ordinary differential equations

$$\frac{dx}{dt} = \mathrm{grad}\, u$$

$$x(0) = x_0$$

$$(3.9)$$

where $\underline{x}=\underline{x}(t)$ is a parametric representation of the given stream-line. Since grad u is only known to a given degree of accuracy for $\underline{x}$ in $\mathbb{R}^3 \backslash B$, (3.9) will only allow us to construct the stream-lines lying in the exterior of B. However, we do know the volume of D from (2.19), and hence by extrapolation we can determine an approximation to the shape of D. Note that by sending in plane waves from different directions we can also determine the streamlines corresponding to a fluid flowing past D from different directions and this can be used to refine the initial estimate for the shape of D.

## IV. Passing Remarks on the Dirichlet Problem

We noted in the Introduction that for the Dirichlet problem the solution of the static potential problem corresponding to a single incoming plane wave is the conductor potential and hence, in contrast to the Neumann problem, no new information is gained by sending in plane waves from different directions. This particular problem can be overcome if two plane waves are transmitted from opposite directions, e.g. $e^{ikx_1}$ and $e^{-ikx_1}$, and the corresponding solutions of the scattering problems added together in such a way that the combined incident field is $\frac{1}{k}\sin kx_1$. The analysis of [4] can then be used to show that the corresponding static potential problem in this case is

$$u_o(\underline{x}) = x_1 + u_o^S(\underline{x}) \quad \text{in } \mathbb{R}^3 \backslash D \tag{4.1a}$$

$$\Delta_3 u_o = 0 \quad \text{in } \mathbb{R}^3 \bar{D} \tag{4.1b}$$

$$u_o(\underline{x}) = 0 \quad \text{for } \underline{x} \in \partial D \tag{4.1c}$$

$$\lim_{r \to \infty} u_o^S(\underline{x}) = 0 \quad . \tag{4.1d}$$

The boundary of the obstacle is again a level surface and except for this boundary the level surfaces change if the plane waves are sent in from different directions than along the $x_1$ axis. Hence if this is done more information can be obtained on the shape of D.

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The low frequency inverse scattering problem in $R^3$ is studied subject to Neumann boundary conditions and acoustic wave propagation.

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